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# Exact Treatment of Mode Locking for a Piecewise Linear Map

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A piecewise linear map with one discontinuity is studied by analytic means in the two-dimensional parameter space. When the slope of the map is less than unity, periodic orbits are present, and we give the precise symbolic dynamic classification of these. The localization of the periodic domains in parameter space is given by closed expressions. The winding number forms a devil's terrace, a two-dimensional function whose cross sections are complete devils's staircases. In such a cross section the complementary set to the periodic intervals is a Cantor set with dimension $D = 0$.

**KEY WORDS:** Nonlinear dynamic systems; circle map; symbolic dynamics; mode locking; fractal dimension.

## 1. INTRODUCTION

In studying the transition from quasiperiodicity to chaos,[1-3] recent interest has focused on so-called circle maps, one-dimensional maps of the circle onto itself. The prototypical mapping, representative for dissipative dynamical systems,[4-6] is the sine map

$$\theta_{n+1} = f(\theta_n)(\text{mod } 1); \qquad f(\theta) = \theta + \Omega - (K/2\pi)\sin(2\pi\theta) \tag{1}$$

Several physical systems can be modeled by circle maps of this kind.[5,6] When the map is invertible ($K \leqslant 1$ for the sine map), the variable converges

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to a quasiperiodic or periodic series. If the period length in the latter case is $Q$, we have

$$f^{Q}(\theta_n) = \theta_n + P \tag{2}$$

where the left-hand side denotes $Q$ times iteration of $f$. The ratio between the integers $P$ and $Q$, $w = P/Q$, is the winding number.

By numerical methods one finds for the periodic orbits stability intervals on the $\Omega$ axis. These mode-locked intervals increase in size when the nonlinearity ($K$) is increased, forming mode-locked domains, also called Arnol'd tongues, in the ($\Omega$, $K$) plane. When the circle map passes from invertibility to noninvertibility ($K = 1$ for the sine map) the measure of the quasiperiodic intervals shrinks to zero, with a fractal dimension, numerically determined to be $D \simeq 0.87$ when the critical map exhibits an inflection point of order three.[4-6]

A different kind of circle map is useful for another class of physical systems, namely nonlinear systems with a limit cycle, driven by an external periodic force. In the fast-relaxation limit the Poincaré map of a two-dimensional system is a circle map.[7,8] The characteristic feature of these maps is that in the critical case when the external force is able to displace the oscillator onto the unstable point within the limit cycle, the map will have a point of discontinuity. For a prototype model of this kind[8,9] the map has constant slope when this happens, and takes the form shown in Fig. 1:

$$x_{n+1} = f(x_n) = \begin{cases} K(x_n + 2\beta), & -1 \leqslant x_n < 1 - 2\beta \\ K(x_n + 2\beta - 2), & 1 - 2\beta < x_n \leqslant 1 \end{cases} \tag{3}$$

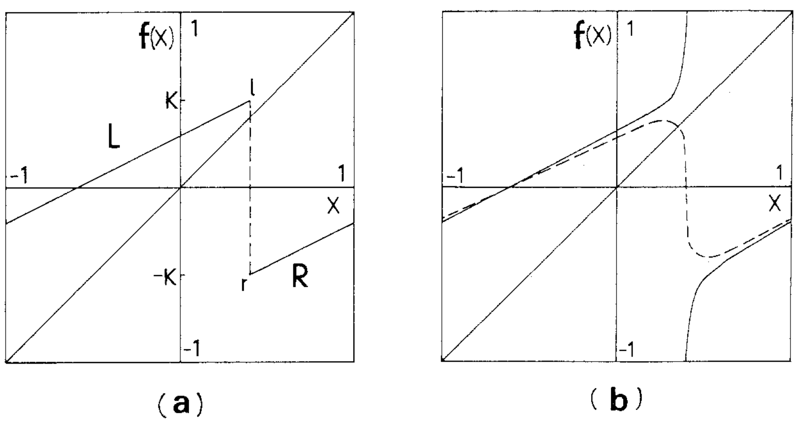


Fig. 1. (a) The piecewise linear map (3), shown for $K = 0.5$ and $\beta = 0.3$. (b) In the model of Ref. 8, the piecewise linear map emerges as the limit of a continuous periodic function on the circle $(-1, 1)$. The piecewise linear map is a critical borderline case between (—) an invertible map and (– –) a noninvertible map.

where $\beta$ represents the ratio between the frequency of the external force and the internal limit cycle frequency.

In this paper we study the properties of the piecewise linear map (3) in the parameter space $0 \leqslant \beta \leqslant 1$ and $0 < K \leqslant 1$ ($|K| > 1$ always corresponds to chaotic behavior, while $-1 < K \leqslant 0$ always yields trivial fixed-point behavior). The motivation for our investigation is threefold: (1) It is interesting to study a circle map that is not in the sine circle map class, (2) it is possible to obtain exact and explicit results, and (3) the results are essential for understanding the complicated general behavior of the prototype oscillator model of Ref. 8.

The mapping (3) has a left branch $L$ and a right branch $R$. Note that we have not yet defined the mapping function at the point of discontinuity, i.e., for

$$x = 1 - 2\beta \equiv \tau \tag{4}$$

We could define $f(\tau)$ as $K$, corresponding to the right limit point of branch $L$ (called $l$ on Fig. 1a), or as $-K$, corresponding to the left limit point of branch $R$ (denoted $r$ on Fig. 1a). It will be relatively unimportant which definition one chooses; the only difference in the final results is that in one case the left boundary of a mode-locked domain will belong to the domain, while in the other case the right boundary will.

Each periodic orbit can be characterized by a *word*, a finite sequence of $L$'s and $R$'s where each letter signifies whether the iterated point visits branch $L$ or branch $R$ of the map. In this symbolic dynamics the word length $Q$ obviously equals the period of the orbit. Denoting the number of $R$'s in the word by $P$, we may define the winding number as

$$w = P/Q \tag{5}$$

From Fig. 1b it is clear that this is equivalent to the usual definition of the winding number.

The special orbits that visit the singular point $x = \tau$ we call *superstable.*

This paper is organized as follows. Section 2 is devoted to the symbolic dynamic classification. We show how the words for periodic orbits can be determined for a given winding number $w$. In Section 3 we determine the boundaries of the mode-locking domains. The resulting structure of parameter space is shown in Fig. 2. For a given $K$ the total measure of the mode-locking intervals is calculated in Section 4 and shown to be unity when $K < 1$. The devil's staircase[10] $w(\beta)$ is thus everywhere complete for all $0 < K < 1$. In Section 5 the dimension of the complementary set of $\beta$ values is shown to be zero, in contrast to the nonzero fractal dimension for critical maps of the sine circle map type.

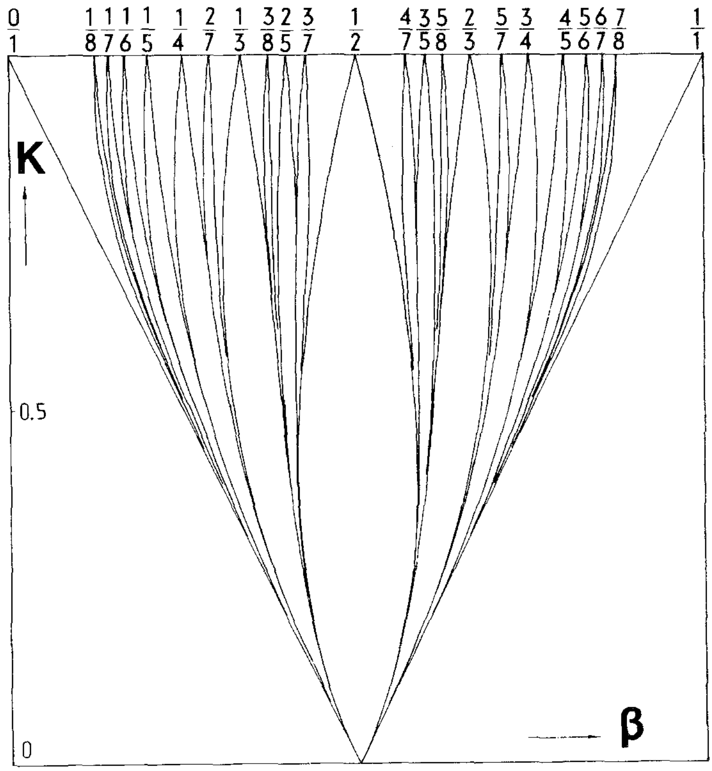


Fig. 2. Arnol'd tongues for periods $Q \leqslant 8$. The fractions on top are the corresponding winding numbers $w$. Fixed-point behavior, $w = 0/1$ and $1/1$, is restricted to the two large triangular domains.

## 2. SYMBOLIC DYNAMICS

The word $W$ of a periodic orbit is defined only up to cyclic permutations. To make it precise, we define $W$ using the superstable orbit that starts at the endpoint $r$ of branch $R$. Figure 3 shows the superstable orbit with $W = RL^3RL^2$, corresponding to a winding number $w = 2/7$.

When $\beta$ increases, the mapping function is displaced horizontally toward the left and it is clear from Fig. 3 that the periodic orbit will change continuously until it finally visits the endpoint $l$ of the left branch. This superstable orbit corresponds to the maximum value of $\beta$, i.e., the right endpoint of the periodic interval. Similarly, the superstable orbit starting from $r$ corresponds to the minimum value of $\beta$. The actual numerical values are calculated in the next section. The word $W$ can be used to characterize the orbit throughout the interval.

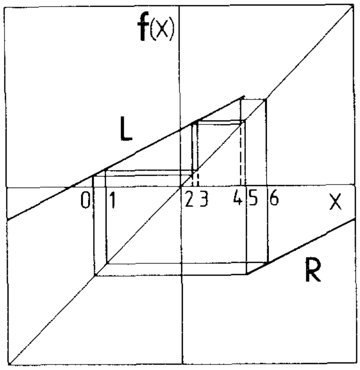


Fig. 3. The superstable orbit $RL^3RL^2$. Here $K=\frac{1}{2}$ and $\beta=161/508=0.316929$. The numbers denote the arguments $z_k$, Eq. (8), in the $x(z_k)$.

For $\beta<\frac{1}{2}$, $x_n>\tau$ implies $x_{n+1}<\tau$, and the branch $R$ will therefore never be visited two times in succession. Similarly, the branch $L$ will not be visited two times in succession for $\beta>\frac{1}{2}$. We may, however, restrict ourselves to $\beta<\frac{1}{2}$, since the mapping (3) is invariant under the symmetry operation $\beta\to 1-\beta$, $x\to -x$. Consequently, for $\beta<\frac{1}{2}$ the word for a winding number $P/Q$ takes the form

$$W=RL^{l_1}RL^{l_2}\cdots RL^{l_P}; \qquad \sum_{i=1}^{P}(l_i+1)=Q \tag{6}$$

To determine the word completely, we note the following two simple properties of the map: (i) Relative ordering is preserved in the sense that when two points $x<x'$ on one branch ($R$ or $L$) both map onto the same branch (not necessarily the original one), then $f(x)<f(x')$. (ii) When a point $x$ on $R$ and a point $x'$ on $L$ both map onto $L$, then $f(x)<f(x')$.

Ordering the $Q$ points of the superstable orbit with winding number $P/Q$ according to magnitude,

$$x(0)<x(1)<\cdots<x(Q-1) \tag{7}$$

the $P$ orbit points on $R$ have the highest values. From property (ii) above it follows that the $P$ lowest values in (7) have the $P$ points on $R$ as preimages. Furthermore, the $P$ lowest points map into the next lowest group of $P$ orbit points on $L$, etc., until all $Q-P$ orbit points on $L$ are exhausted and mapping onto $R$ takes place. The starting point $r$ maps into $x(0)$, $x(P)$, $x(2P)$, etc., until branch $R$ is reached; then the orbit switches back to $L$.

In short, the $(k+1)$th image of $r$ is located at $x(z_k)$, where

$$z_k = kP(\text{mod } Q) \tag{8}$$

From the sequence $z_0, z_1, \ldots, z_{Q-1}$, which obviously starts with $z_0 = 0$, the word $W(P/Q)$ follows at once, since $z_n < z_{n-1}$ implies switching from $R$ to $L$. The first letter is always $R$. The $(n+1)$th letter, $n \geqslant 1$, is $R$ if $z_n < z_{n-1}$, otherwise it is $L$.

As a simple example, take $w = 2/7$. The $z_k$ sequence equals 0, 2, 4, 6, 1, 3, 5, and the word

$$W(2/7) = RLLLRLL = RL^3RL^2 \tag{9}$$

results (see Fig. 3).

We note in passing that an alternative algorithm for obtaining the words uses continued fractions. Let

$$\frac{P}{Q} = \cfrac{1}{N_1 + \cfrac{1}{N_2 + \cfrac{1}{N_3 + \cdots}}} \tag{10}$$

be the continued fraction representation of the winding number, and let $P_1/Q_1$, $P_2/Q_2$, etc., be the successive approximants. Then the words corresponding to $P/Q$, as well as to the approximants $w_i = P_i/Q_i$, are given by

$$\begin{aligned} &W_1 = RL^{N_1-1} \\ &W_2 = W_1 L W_1^{N_2-1} \\ &W_i = W_{i-1}^{N_i} W_{i-2} \qquad \text{for odd} \quad i \geqslant 3 \\ &W_i = W_{i-2} W_{i-1}^{N_i} \qquad \text{for even} \quad i \geqslant 4 \end{aligned} \tag{11}$$

We will not use this representation, and omit the proof.[9]

For a given winding number, $\beta$ can increase until the orbit reaches the endpoint $l$ of $L$. At this $\beta$ value, $\beta_+$, the orbit is again superstable, by definition. Instead of describing the orbit with the same word $W$ as before, it will be useful to characterize it by a word $\hat{W}$ corresponding to using $l$ as the starting point. It is interesting that $\hat{W}$, trivially a cyclic permutation of $W$, is in fact *identical* to $W$, except for a transposition of the first two letters. That is, if

$$W = RLX$$

with an unspecified sequence $X$ of $R$'s and $L$'s, then

$$\hat{W} = LRX$$

To prove this, note that the orbit now is $x(y_0)$, $x(y_1)$,..., with $y_0 = Q-1$, $y_1 = P-1$, $y_k = kP-1 \pmod{Q}$ for $k \geqslant 1$. In other words, for $k>0$, $z_k = y_k+1$, and $z_{k+1} - z_k = y_{k+1} - y_k$. The words will therefore be identical, barring the two first letters, which are trivially seen to be $RL$ and $LR$, respectively.

As will be shown in the next section, this result will be useful for determining the widths of the Arnol'd tongues.

## 3. THE MODE-LOCKING REGIONS

The simplest periodic orbits are fixed points, $Q=1$. As long as the branch $L$ intersects the diagonal $x_{n+1} = x_n$, the periodic orbit with word $L$ occurs, with winding number 0/1. This fixed-point solution occurs as long as $K < \tau = 1 - 2\beta$. Hence the region with winding number $w = 0/1$ is given by

$$0 \leqslant \beta \leqslant \tfrac{1}{2}(1-K) \tag{12}$$

Similarly, the fixed point with word $R$ is present when the branch $R$ intersects the diagonal. This occurs when $\tau \leqslant -K$. Thus, winding number $w = 1/1$ corresponds to the region

$$\tfrac{1}{2}(1+K) \leqslant \beta \leqslant 1 \tag{13}$$

For intermediate values, $\frac{1}{2}(1-K) < \beta < \frac{1}{2}(1+K)$, longer periods exist. For example, periodic orbits $RL$ (or $LR$) occur when the equation

$$x = f(f(x))$$

possesses a solution. And once again the critical values of $\beta$, which determine the boundaries of the domain with winding number $\frac{1}{2}$, correspond to the two superstable orbits. The superstable orbit with word $W = RL$ is determined by $\tau = f_L(f_R(\tau))$, or equivalently,

$$f_R^{-1}(f_L^{-1}(\tau)) = \tau \tag{14}$$

The inverse map of the two branches of $f(x)$ can be expressed as

$$f_R^{-1}(x) = R(x) - 2\beta, \qquad f_L^{-1}(x) = L(x) - 2\beta \tag{15}$$

with

$$R(x) = K^{-1}x + 2, \qquad L(x) = K^{-1}x \tag{16}$$

For the superstable orbit $RL$, (13) yields

$$\beta_- = \tfrac{1}{2}(1+K^2)/(1+K) \tag{17}$$

and the superstable orbit $LR$ is similarly found to occur for

$$\beta_+ = \tfrac{1}{2}(1+2K-K^2)/(1+K) \tag{18}$$

The winding number $w = \frac{1}{2}$ is therefore restricted to the region

$$\tfrac{1}{2}(1+K^2)/(1+K) < \beta < \tfrac{1}{2}(1+2K-K^2)/(1+K) \tag{19}$$

In the general case it is clear that the equation for determining a superstable orbit $XY \cdots Z$ is

$$f_{\bar{X}}^{-1}(f_{\bar{Y}}^{-1}(\cdots f_{\bar{Z}}^{-1}(\tau)\cdots)) = \tau \tag{20}$$

where the subscripts form the word of the orbit. Equation (14) is a special case. Now we insert (15), and use the following properties:

$$R(a+b) = R(a) + K^{-1}b, \qquad L(a+b) = L(a) + K^{-1}b \tag{21}$$

Hence

$$f_{\bar{B}}^{-1}(\tau) = B(1-2\beta) - 2\beta = B(1) - 2K^{-1} - 2\beta \tag{22}$$

$$f_{\bar{A}}^{-1}(f_{\bar{B}}^{-1}(\tau)) = AB(1) - 2\beta K^{-2} - 2\beta K^{-1} - 2\beta \tag{23}$$

etc. We use the notation $AB(1)$ rather than $A(B(1))$. Here $A$ and $B$ represent either $R$ or $L$. The general structure of (20) is now clear. If the word of the superstable orbit is $W$, then (20) reduces to

$$W(1) - 2\beta \sum_{n=0}^{Q} K^{-n} = 1 - 2\beta \tag{24}$$

or

$$\beta = \frac{1-K}{2K^{-Q}-2}\,[W(1)-1] \tag{25}$$

The function $W$ in (25) is composed of the two functions (16) in accordance with the word structure. $Q$ is the period.

The commutator of the operators $L$ and $R$ is a *number*,

$$LR - RL = 2K^{-1} - 2 \tag{26}$$

This makes it very easy to find the total width of the interval $(\beta_-, \beta_+)$ with winding number $P/Q$,

$$\Delta\beta = \beta_+ - \beta_- = \frac{1-K}{2K^{-Q}-2}\,[W(1) - \hat{W}(1)] \tag{27}$$

Since the $W$, $\hat{W}$ of the two superstable orbits are identical except for the transposition $LR \to RL$ of the two first letters, (26) yields immediately

$$\Delta\beta = (1-K)^2 K^{-1}/(K^{-Q}-1) \tag{28}$$

The width of an Arnol'd tongue is therefore independent of the winding number numerator $P$ in our case.

Equation (28) is all that is needed for a determination of the total measure of all periodic domains (see the next section). The remaining part of this section is devoted to a determination of explicit expressions for the left and right boundaries $\beta_{\pm}(K)$ of the region with winding number $P/Q$.

We have to evaluate $W(1)$, using the functions (16), $L(x) = K^{-1}x$, and $R(x) = K^{-1}x + 2$. For the orbit (word) with the minimum number of $R$'s, namely $W = RL^{Q-1}$, we have trivially

$$W(1) = K^{-Q} + 2 \tag{29}$$

Other words with the same period length result when some of the $L$'s are replaced by $R$'s. By such replacements extra terms are added, since $R(x) = L(x) + 2$. According to the symbolic dynamic algorithm of Section 2, the $(n+1)$th letter is $R$ if, and only if,

$$[nP/Q] - [(n-1)P/Q] = 1 \tag{30}$$

where $[x]$ denotes the integer part of $x$. Adding up the contributions from these additional $R$'s, we obtain

$$w(1) = K^{-Q} + 2 + 2 \sum_{n=2}^{Q-1} \{[nP/Q] - [(n-1)P/Q]\} K^{-n} \tag{31}$$

Insertion of (31) into (25) yields an explicit expression for the left boundary $\beta_-$ of the $P/Q$ domain. The right boundary $\beta_+ = \beta_- + \Delta\beta$ follows then from (28). The result can be rewritten as

$$\beta_{\pm} = \frac{1}{2} - \frac{1}{2} K + \frac{1-K}{1-K^Q} (P - 1 + K^{Q-1/2 \mp 1/2}) - \frac{(1-K)^2}{1-K^Q} \sum_{n=1}^{Q-1} \left[\frac{nP}{Q}\right] K^{Q-n-1} \tag{32}$$

## 4. THE MEASURE OF THE PERIODIC REGIONS

We have seen that the widths $\Delta\beta$ of the periodic domains are the same for all winding numbers $P/Q$ with a given period length $Q$. For a given slope $K$, the total measure $M$ of all periodic domains is therefore

$$M(K) = (1-K)^2 K^{-1} \sum_{Q=1}^{\infty} \phi(Q)/(K^{-Q}-1) \tag{33}$$

Here $\phi(Q)$, Euler's $\phi$-function,[11] equals the number of positive integer less than $Q$ and relative prime to $Q$. Given $Q$, $\phi(Q)$ is the number of possible values for the numerator $P$ of the winding number.

The Liouville formula[11] for $|K| < 1$,

$$\sum_{Q=1}^{\infty} \frac{\phi(Q)\, K^Q}{1 - K^Q} = K(1-K)^{-2} \tag{34}$$

yields at once $M = 1$ for every $K < 1$. For $K = 1$ the map (3) is the simple shift map, and periodic orbits require $\beta$ rational. In conclusion,

$$M(K) = \begin{cases} 1 & \text{for} \quad K < 1 \\ 0 & \text{for} \quad K = 1 \end{cases} \tag{35}$$

The winding number $w$ locks in at every rational number $P/Q$, and at fixed $K$ the function $w(\beta, K)$ thus forms a devil's staircase.[10] By (35) the staircase is complete for $K < 1$. In the present case we have an explicit expression for the position of the steps, Eq. (32). As a function of *both* parameters the function $w(\beta, K)$ forms a "devil's terrace" (Fig. 4).

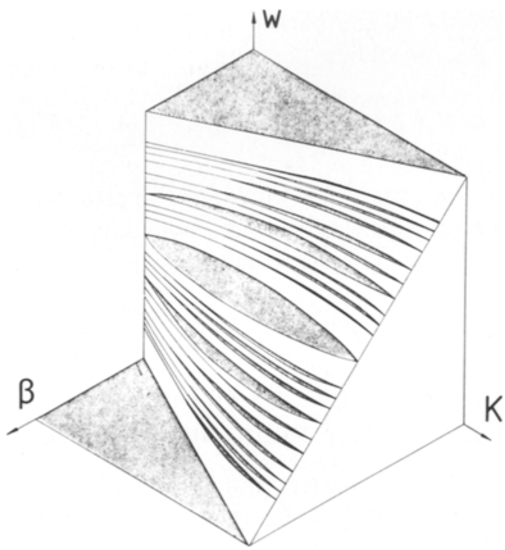


Fig. 4. The devil's terrace. The winding number $w = P/Q$ as function of the map parameters $K$ and $\beta$. (Only plateaus with $Q \leqslant 8$ are shown.)

The widths of the periodic regions—the Arnol'd tongues—are zero at $K=1$. When $K$ is lowered, the widths increase linearly in the beginning:

$$\Delta\beta = Q^{-1}(1-K) + O(1-K)^2 \tag{36}$$

For $Q>1$ the width of an Arnol'd tongue goes through a maximum $\Delta_m$ at a value $K=K_m$, which is close to 1 when the period is large. In this way it is possible to have a total measure independent of $K$. It is easy to show that

$$K_m = 1 - cQ^{-1} + O(Q^{-2}) \tag{37}$$

and

$$\Delta_m = \hat{c}Q^{-2} + O(Q^{-3}) \tag{38}$$

where $c=1.5936...$ is the relevant root of $c+2e^{-c}=2$, and $\hat{c}=c^2/(e^c-1)=0.6476...$ .

## 5. DIMENSION OF THE COMPLEMENTARY SET

We have just shown that the periodic intervals take the whole measure on the $\beta$ axis for $K<1$. The complementary set of $\beta$ values (not empty!) form a Cantor set, and it is interesting to assess its fractal dimension $D$. We want to compare with the corresponding dimension $D=0.87$ of the sine circle map, and consequently use the same definition of $D$ as Ref. 5.

Let the resolution be defined by a minimum step length $\varepsilon$ in the devil's staircase, and let $\bar{M}(\varepsilon)$ be the measure of the remaining set when all steps larger than $\varepsilon$ are taken out. Measured in units of the minimum step $N(\varepsilon)=\bar{M}(\varepsilon)/\varepsilon$, the small-$\varepsilon$ behavior, $N(\varepsilon)\sim\varepsilon^{-D}$, or

$$D = \lim_{\varepsilon\to 0} D(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{|\ln \varepsilon|} \tag{39}$$

defines a fractal dimension $D$.

The minimum step length is connected to a maximum period $G$ through (28),

$$\varepsilon = K^{-1}(1-K)^2/(K^{-G}-1) \tag{40}$$

and the measure of the remaining set equals

$$\bar{M}(\varepsilon) = K^{-1}(1-K)^2 \sum_{Q=G+1}^{\infty} \phi(Q)/(K^{-Q}-1) \tag{41}$$

Hence

$$N(\varepsilon) = \sum_{Q=G+1}^{\infty} \frac{1-K^G}{1-K^Q}\,\phi(Q)\,K^{Q-G} \tag{42}$$

Using that $\phi(Q) \sim 6\pi^{-2}Q$ "in the mean,"[11] it would follow that

$$\lim_{G \to \infty} \frac{N(\varepsilon)}{G} = \frac{6K}{\pi^2(1-K)} \tag{43}$$

For our purpose it suffices to use the inequalities $\phi(Q) \leqslant Q$ and $1 - K^G < 1 - K^Q$ in (42) to produce the inequality

$$N(\varepsilon) < G\frac{K}{1-K} + \frac{K}{(1-K)^2} \tag{44}$$

to show that $N(\varepsilon)$ increases at most linearly with $G$. [It is also easy to show that $N(\varepsilon) \geqslant 1$.] From (40) it follows that

$$\lim_{G \to \infty} \frac{\ln \varepsilon}{G} = \ln K \tag{45}$$

Equations (44) and (45) imply that asymptotically

$$D(\varepsilon) \lesssim \ln G \, |G \ln K|^{-1} \tag{46}$$

and consequently

$$D = 0 \tag{47}$$

for $K < 1$. Alstrøm[12] reports that the completeness and the result that $D = 0$ have already been calculated by B. Søderberg (not published). The convergence of $D(\varepsilon)$ to $D$ is extremely slow. As an example, numerical evaluation gives $D(1.5 \times 10^{-11}) = 0.12$ for $K = \frac{1}{2}$.

An interesting question is the universality of the result $D = 0$. A likely possibility is that any map $x = f(x)$ with positive slope less than unity, except for one discontinuity, will be in the $D = 0$ universality class.

## REFERENCES


1. S. J. Shenker, *Physica* **5D**:405 (1982).
2. M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, *Physica* **5D**:370 (1982).
3. D. Rand, S. Ostlund, J. Sethna, and E. Siggja, *Phys. Rev. Lett.* **69**:132 (1982).
4. M. H. Jensen, P. Bak, and T. Bohr, *Phys. Rev. Lett.* **50**:1637 (1983).
5. M. H. Jensen, P. Bak, and T. Bohr, *Phys. Rev. A* **30**:1960 (1984).
6. M. H. Jensen, P. Bak, and T. Bohr, *Phys. Rev. A* **30**:1970 (1984).
7. L. Glass and R. Perez, *Phys. Rev. Lett.* **48**:1772 (1982).
8. E. J. Ding, *Phys. Rev. A* **34**:3547 (1986).
9. E. J. Ding, *Phys. Rev. A* (1987), to appear.
10. S. Aubry, in *Solitons and Condensed Matter Physics*, A. R. Bishop and T. Schneider, eds. (Springer, Berlin, 1979).
11. L. E. Dickson, *History of the Theory of Numbers*, Vol. 1 (Chelsea, New York, 1966); T. Nagell, *Introduction to Number Theory* (Chelsea, New York, 1981).
12. P. Alstrøm, *Commun. Math. Phys.* **104**:581 (1986).